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WRONSKIAN AND IT'S PROPERTIES

by

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Basic Theory of Differential Equations →

A differential Equation is said to be linear if the unknown function and all of its derivatives occuring in the equation occur only in the first degree and are not multiplied together.

e.g → (1) $\frac{d^2y}{dx^2} + 2y = 0$ (2) $\frac{dy}{dx} = \cos x$ are linear differential Equations

(3) $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^2 = 0$ is non-linear differential Equation.

General Linear Equation → The general linear differential Equation of order n is

$$a_0 \frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = F(x),$$

$a_0 \neq 0$. - - - - - - - - (1).

Where $a_0, a_1, a_2, \ldots a_n$ and F are Continuous real functions of x only over an open Interval I.

Note: 1 If $F = 0$ over I, then it is called Homogeneous.

(2) If $F \neq 0$ over I, then (1) is called Non-Homogenous differential Equation.

For $n = 2$

Equation (1) reduces to the second order non-Homogenous Linear Equation.

$$\boxed{a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x)}$$

& Corresponding second order Homogenous Equation is

$$\boxed{a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0} \; ; \; a_0(x) \neq 0$$

Existence and Uniqueness THEOREM →

If $a_0(x), a_1(x), a_2(x), \ldots\ldots a_n(x)$ and F are all continuous functions of x on a real interval $a \le x \le b$ and $a_0(x) \ne 0$ for each x on $a \le x \le b$.

Let $c_0, c_1, c_2, \ldots c_{n-1}$ are n arbitrary real constants and x_0 is any point of the interval $a \le x \le b$.

then there Exists one and only one function say $f(x)$ (unique function) of (1) satisfying

$$f(x_0) = c_0;\ f'(x_0) = c_1, \ldots\ldots f^{n-1}(x_0) = c_{n-1}$$

which in some neighbourhood of x_0 (contained in $a \le x \le b$) is solution of the differential Equation.
(i.e this solution $f(x)$ is defined over the interval $a \le x \le b$)

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x).$$

Remarks 1 → The above theorem is an Existence theorem because it says that the initial value problem does have a solution.

(2) It is also known as uniqueness theorem because, it says that there is only one solution

(3) This theorem also applies to an associated Homogenous Equation.

(4) In this Chapter, we shall assume without proof

(5) The Conditions of Existence and uniqueness theorem Can not be further relaxed!
For Example if $a_0(x) = 0$ for some $x \in (a,b)$, then the solution of (1) may not be unique or may not Exist at all.

* Existence and Uniqueness theorem for a second Order linear differential Equation

Consider the general Equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \quad \text{——} (1)$$

where $a_0(x), a_1(x), a_2(x)$ & $F(x)$ are Continuous functions on an interval (a,b) and $a_0(x) \neq 0$ for each $x \in (a,b)$. Let c_1 and c_2 be arbitrary real numbers. $\forall x_0 \in (a,b)$ Then $\exists$ a unique solution $f(x)$ of (1) satisfying $f(x_0) = c_1$ and $f'(x_0) = c_2$. Moreover, this solution $f(x)$ is defined over the interval (a,b)

(Without proof).

Cor. → Let f be a solution of the n^{th} order homogeneous linear differential Equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

such that

$$f(x_0) = 0, f'(x_0) = 0, \cdots f^{n-1}(x_0) = 0$$

where x_0 is a point on the interval (a,b) in which the coefficients $a_0, a_1, \ldots a_{n-1}, a_n$ are all continuous & $a_0(x) \neq 0$ then $\boxed{f(x) = 0}$ $\forall\, x \in (a,b)$.

Linear Combination →

If $f_1, f_2, \cdots f_n$ be n functions defined over an interval I and $C_1, C_2 \cdots C_n$ are n arbitrary constants then the function

$C_1 f_1 + C_2 f_2 + C_3 f_3 + \cdots + C_n f_n$ is called a linear combination of the functions $f_1, f_2, \cdots f_n$ over I.

e.g ① $2e^x + 3e^{2x} + 7e^{5x}$ is a linear Combination of e^x, e^{2x}, e^{5x}

Note *. Any linear Combination of solutions of a linear Homogenous differential Equation is a solution of the Equation.

LINEARLY DEPENDENT SET OF FUNCTIONS →

The functions $f_1, f_2, \dots f_n$ of x are said to be linearly dependent (or L.D) over an interval I if there exists constants $c_1, c_2, \dots c_n$ not all zero such that

$$\boxed{c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0} \text{ for all } x \text{ in } I$$

For Example → The functions $x^2, 3x^2$ are L.D on [0,1] because there Exists constants $c_1 = -3$, $c_2 = 1$ (not all zero) such that

$$(-3)(x^2) + (1)(3x^2) = 0.$$

Linearly Independent → The functions $f_1, f_2, \dots f_n$ of x are said to be linearly independent or (L.I) over an interval I iff there exists constants $c_1, c_2, \dots c_n$ such that the relation

$$\boxed{c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0} \text{ for all } x \in I$$

$$\Rightarrow c_1 = c_2 = \dots = c_n = 0$$

Note *

If the functions $f_1, f_2, \ldots f_n$ are linearly Dependent (L.D), then atleast one of them is a linear combination of the others

WRONSKIAN → Let $f_1, f_2, \ldots f_n$ be n real functions over an interval I, each of which has a derivative of atleast $(n-1)$ order over I, then the determinant

$$\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

denoted by $W(f_1, f_2, \ldots f_n)$ is called Wronskian of $f_1, f_2, \ldots f_n$ over I.

Theorem → Prove that if the **Wronskian** of the functions $f_1, f_2, \ldots f_n$ over an interval I is non-zero, then these functions $f_1, f_2, \ldots f_n$ are linear independent over I.

Proof → Suppose that Wronskian of the functions $f_1, f_2, \ldots f_n$ over an interval I is non-zero.

Now, we have to show that $f_1, f_2, \ldots f_n$ are L.I over I.

For this, Consider the relation $\boxed{c_1 f_1 + c_2 f_2 + \cdots + c_n f(n) = 0}$

where $c_1, c_2, \ldots c_n$ are Constants —— (1)

Differentiating (1) successively $n-1$ times w.r.t 'x', we get

$$c_1 f_1' + c_2 f_2' + \cdots\cdots + c_n f_n' = 0 \quad \text{——(2)}$$

$$c_1 f_1'' + c_2 f_2'' + \cdots\cdots + c_n f_n'' = 0 \quad \text{——(3)}$$

$$\cdots\cdots\cdots$$

$$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \cdots\cdots + c_n f_n^{(n-1)} = 0 \quad \text{——(n)}$$

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These n equations can be written as

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now, we know that the matrix Equation $AX=0$ has a trivial solution i.e ($c_1 = c_2 = \cdots = c_n = 0$) iff $|A| \neq 0$

$$\therefore \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} \neq 0$$

which is a $W(f_1, f_2, \cdots f_n) \neq 0$

Since we have already supposed that Wronskian $W(f_1, f_2, \cdots f_n) \neq 0$.

$\therefore c_1 = c_2 = \cdots = c_n = 0$

$\therefore$ the functions $f_1, f_2, \cdots f_n$ are linearly Independent over I.

Note * $\rightarrow$ From the above theorem, it follows that non-Vanishing of the Wronskian over an interval I is the sufficient condition that the functions are L.I over I.

Theorem 2 → If $f_1, f_2, \ldots, f_n$ are n linearly independent solutions of n^{th} order homogeneous linear differential Equation

$$L_n(y) = a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

Then every solution of $L_n(y) = 0$ can be expressed as a suitable linear combination $c_1f_1 + c_2f_2 + \cdots + c_nf_n$ of the n linearly independent solutions.

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ILLUSTRATIVE EXAMPLES

Example 1 → Show by Wronskian that x, x^3, x^4 are linearly independent if x is non-zero.

Sol: Let $f_1(x) = x$, $f_2(x) = x^3$, $f_3(x) = x^4$.

$\therefore f_1'(x) = 1$; $f_2'(x) = 3x^2$; $f_3'(x) = 4x^3$

$f_1''(x) = 0$; $f_2''(x) = 6x$; $f_3''(x) = 12x^2$.

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} x & x^3 & x^4 \\ 1 & 3x^2 & 4x^3 \\ 0 & 6x & 12x^2 \end{vmatrix}$$

$$= x \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & 3x^2 & 4x^3 \\ 0 & 6x & 12x^2 \end{vmatrix}.$$

$$= x \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & 2x^2 & 3x^3 \\ 0 & 6x & 12x^2 \end{vmatrix}$$ by $R_2 \to R_2 - R_1$

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$$= x \begin{vmatrix} 2x^2 & 3x^3 \\ 6x & 12x^2 \end{vmatrix} = x\left(24x^4 - 18x^4\right) = 6x^5 \neq 0 \text{ as } x \neq 0.$$

$\therefore W(f_1, f_2, f_3) \neq 0$

$\boxed{\therefore f_1, f_2, f_3 \text{ are L.I}}$

<u>Example 2</u> → Show by Wronskian that the following functions are L.I over all reals.

(i) $\sin 2x$, $\cos 2x$

<u>Sol</u>:- Let $f_1(x) = \sin 2x$, $f_2(x) = \cos 2x$

$\therefore f_1'(x) = 2\cos 2x$, $f_2'(x) = -2\sin 2x$.

$$\therefore W(f_1, f_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2\sin^2 2x - 2\cos^2 2x$$

$$= -2\left[\sin^2 2x + \cos^2 2x\right] = -2(1) = -2 \neq 0 \text{ for all } x$$

$\therefore W(f_1, f_2) \neq 0$.

$\therefore f_1 + f_2$ are L.I over reals.

Example 3 → Show by Wronskian that the following functions are linearly independent over all reals.

(i) $e^{ax}\cos bx,\ e^{ax}\sin bx$.

Sol: Let $f_1(x) = e^{ax}\cos bx$; $f_2(x) = e^{ax}\sin bx$.

$\& \ f_1'(x) = e^{ax}(-\sin bx).b + a.e^{ax}\cos bx.$

$= e^{ax}\left[-b\sin bx + a\cos bx\right]$

$= e^{ax}(a\cos bx - b\sin bx).$

$f_2'(x) = e^{ax}\cos bx.b + a e^{ax}\sin bx$

$= e^{ax}\left[b\cos bx + a\sin bx\right]$

$$\therefore W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ e^{ax}(a\cos bx - b\sin bx) & e^{ax}(b\cos bx + a\sin bx) \end{vmatrix}$$

$= e^{2ax}\left[b\cos^2 bx + a\cos bx\sin bx\right] - e^{2ax}\left[a\cos bx\sin bx - b\sin^2 bx\right].$

$= e^{2ax}\left(b\cos^2 bx + a\cancel{\cos bx\sin bx} - a\cancel{\cos bx\sin bx} + b\sin^2 bx\right)$

$= e^{2ax}\left(b\cos^2 bx + b\sin^2 bx\right)$

$= be^{2ax}\left(\cos^2 bx + \sin^2 bx\right) = be^{2ax}(1) = be^{2ax} \neq 0$ for all real x.

$\therefore W(f_1, f_2) \neq 0$

$\Rightarrow f_1, f_2$ are L.I over reals.

(ii) e^{ax}, xe^{ax}

Sol: Let $f_1(x) = e^{ax}$; $f_2(x) = xe^{ax}$

$\therefore f_1'(x) = ae^{ax}$; $f_2'(x) = axe^{ax} + e^{ax}$

Now, $W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & (axe^{ax} + e^{ax}) \end{vmatrix}$

$= \begin{vmatrix} e^{ax} & xe^{ax} \\ ae^{ax} & e^{ax}(ax+1) \end{vmatrix}$

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$= e^{2ax}(ax+1) - axe^{2ax} = e^{2ax}(\cancel{ax}+1 - \cancel{ax})$

$= e^{2ax}(1) = e^{2ax} \neq 0 \ \forall \ x \in \mathbb{R}$.

$\therefore f_1, f_2$ are L.I over reals.

Example 4 → Show that the function $2x, 3x$ are not L.I.

Sol: We can show that $2x, 3x$ are not L.I over reals if there exists constants C_1, C_2 not all zero s.t

$$\boxed{C_1(2x) + C_2(3x) = 0}$$

Taking $\boxed{C_1 = -3, \ C_2 = 2}$ we get the above relation satisfied. $\therefore C_1 \neq 0, C_2 \neq 0$ not both zero but $C_1(2x) + C_2(3x) = 0$

$\therefore$ given functions $2x, 3x$ are not L.I.

Example 5 → Show that $\sin x$, $\cos x$, $2\sin x + 3\cos x$ are linearly dependent.

Solution:→ The given functions are $\sin x$, $\cos x$, $2\sin x + 3\cos x$ & these functions f_1, f_2, f_3 will be linearly dependent over all reals if ∃ constants c_1, c_2, c_3 not all zero s.t

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \text{ for all reals.}$$

i.e $c_1 \sin x + c_2 \cos x + c_3 (2\sin x + 3\cos x) = 0$ for all reals x

Taking $\boxed{c_1 = -2,\ c_2 = -3,\ c_3 = 1}$ we find that

$$(-2)\sin x + (-3)\cos x + (1)(2\sin x + 3\cos x) = 0$$

∴ c_1, c_2, c_3 not all zero but $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

∴ f_1, f_2, f_3 are linearly Dependent

Example 6 → Show that the functions $e^{2x}\sin 3x$, $e^{2x}\cos 3x$, $e^{2x}(2\sin 3x + 5\cos 3x)$ are linearly Dependent.

Sol: Here $f_1(x) = e^{2x}\sin 3x$; $f_2(x) = e^{2x}\cos 3x$

$f_3(x) = e^{2x}(2\sin 3x + 5\cos 3x)$

∴ $f_1(x), f_2(x)$ & $f_3(x)$ are L.D if ∃ constants c_1, c_2, c_3 not all zero s.t $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

i.e $c_1 e^{2x}\sin 3x + c_2 e^{2x}\cos 3x + c_3[2e^{2x}\sin 3x + 5e^{2x}\cos 3x] = 0$

Example 7 → Show that $1, \sin^2 x, \cos^2 x$ are L.D

Sol: Here $f_1(x) = 1$; $f_2(x) = \sin^2 x$; $f_3(x) = \cos^2 x$.

or $f_1(x) = 1$; $f_2(x) = \sin^2 x$; $f_3(x) = 1 - \sin^2 x$.

Now the functions $f_1(x), f_2(x), f_3(x)$ are L.D if $\exists$ c_1, c_2, c_3 not all zero s.t

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$

i.e $c_1(1) + c_2 \sin^2 x + c_3(1 - \sin^2 x) = 0$

or $c_1 + c_2 \sin^2 x + c_3 - c_3 \sin^2 x = 0$

or $\boxed{(c_1 + c_3) + (c_2 - c_3)\sin^2 x = 0}$

Taking $\boxed{c_1 = -1,\ c_2 = 1,\ c_3 = 1}$ we find the above relation satisfied.

$\therefore f_1, f_2, f_3$ are L.D over reals.

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Example 8 → $e^x, e^{-x}, \cosh x$ are L.D

Sol: Here $f_1(x) = e^x$; $f_2(x) = e^{-x}$; $f_3(x) = \cosh x$

or $f_1(x) = e^x$; $f_2(x) = e^{-x}$; $f_3(x) = \dfrac{e^x + e^{-x}}{2}$

Now these functions f_1, f_2, f_3 are L.D over reals if $\exists$ constants c_1, c_2, c_3 not all zero such that

$$c_1 e^x + c_2 e^{-x} + c_3\left(\frac{e^x}{2} + \frac{e^{-x}}{2}\right) = 0$$

or $(2c_1 + c_3)e^x + (2c_2 + c_3)e^{-x} = 0$

$\boxed{\text{Taking } c_1 = 1,\ c_2 = 1,\ c_3 = -2}$, we find the above

Example → Give an example of functions which are linearly independent yet their Wronskian vanishes on the given interval

Sol: Let $f_1(x) = \begin{cases} x^2 & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$; $f_2(x) = \begin{cases} 0 & ; x \geq 0 \\ x^2 & ; x < 0. \end{cases}$

First we shall show that the functions $f_1(x), f_2(x)$ are L.I. For this, we will show that

$$\boxed{c_1 f_1(x) + c_2 f_2(x) = 0 \quad \text{for all } x \in \mathbb{R}}$$

$\Rightarrow c_1 = c_2 = 0.$

For this purpose, use $x = -1$ & 1.

$f_1(-1) = 0$; $f_2(-1) = 1$

& when $x = 1 \Rightarrow f_1(1) = 1$; $f_2(1) = 0$.

Sub. these values in (1), we get.

$c_1(0) + c_2(1) = 0$

& $c_1(1) + c_2(0) = 0$

we have $c_1 f_1 + c_2 f_2 = 0 \Rightarrow c_1 = c_2 = 0.$

Hence f_1, f_2 are L.I on the given interval.

Next, we will find the Wronskian of the given functions

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<u>Case I.</u> When $x \geq 0$, we have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 2x & 0 \end{vmatrix}$$

$$= (0-0) = 0$$

<u>Case II</u> $\rightarrow$ When $x < 0$, we have

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2 \\ 0 & 2x \end{vmatrix}$$

$$= (0-0) = 0$$

Combining the results of Cases I & II, we get $W(f_1, f_2) = 0$ for all x.

0